A STUDY OF OPERATOR FRAMES IN HILBERT SPACES

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INTRODUCTION

Frame theory is nowadays a fundamental research area in mathematics, computer science, and engineering with many exciting applications in a variety of different fields. It is generally acknowledged that frames were born in 1952 in the work of Duffin and Schaeffer [55]. Its significance for signal processing has been revealed in the pioneering work by Daubechies, Grossman and Meyer in 1986. Since then, frame theory has quickly become the key approach whenever redundant, yet stable, representation of data is required.

In the study of vector spaces, one of the most important concepts is that of a basis. In fact, a basis provides us with an expansion of all vectors in terms of "elementary building blocks" and hereby helps us by reducing many questions concerning general vectors to similar questions concerning only the basis elements. However, the conditions to a basis are very restrictive: we require that the elements are linearly independent, and very often we even want them to be orthogonal with respect to an inner product.

This makes it hard or even impossible to find basis satisfying extra conditions, and this is the reason that one might wish look for a more flexible tool called a frame.

Frame is a generalization of basis. A frame for a vector space equipped with an inner product also allows each element in the space to be written as a linear combination of the elements in the frame, but linear independence between the frame elements is not required. Intuitively, one can think about a frame as a basis to which one has added more elements.

Frame theory has been used to capture significant signal characteristics, which provide numerical stability of reconstruction, and enhance resilience to additive noise.

Several signal processing applications today are based on the use of different transforms. The signals under consideration are written as a linear combination of some predefined set of functions. Orthogonal basis has been used for this purpose. The theory of orthogonal basis for Hilbert space can, however, be generalized to other sequence of functions called as a frame.

Frames are signal representation tools that are redundant, and since they are less constrained than bases, they are used when more flexibility in choosing representation is needed.

The concept of frame theory was introduced in 1952 by R.J. Duffin and A.C. Schaeffer [55] while discussing some problems in the theory of non-harmonic Fourier series. This concept was again taken up in the mid 80's by I. Daubechies [24]. This theory was developed by Peter G. Casazza, Ole Christensen, D. Han and D.R. Larson [44,38, and 9].

D. Han and D.R. Larson [9] developed a number of basic aspects of operator-theoretic approach to frame theory in Hilbert space. Peter G. Casazza [44] presented a tutorial on frame theory and he suggested the major directions of research in frame theory.

John J. Benedetto and Matthew Fickus [27] developed the theory of finite normalized tight frames. Peter G.Casazza [44, 52] has not only given the frame work of a theory but also established new results. Radu V. Balan and Peter G.Casazza [57] are analyzed decomposition of a normalized tight frames and obtained identities for frames.

L.Gavruta [33] has explained history of frames and its applications in different fields. Atomic system of frames is defined and a few equivalent conditions on them were explained. A family of local atomic frames are defined and results on them were obtained.

The notation of K-frames has been introduced by Gavrutha [33] to study the atomic system with respect to a bounded linear operator K in a Separable Hilbert space

H. Results on K-frames have been proved through operator-theoretic results on quotient of bounded operators by G. Ramu and P.Johnson[17].

Fahimeh Arabyani Neyshaburi and Ali Akber Arefijamaal[16] were characterize all duals of a given K- frame and given some approaches for constructing K-frames. Mitra Shamsabadi and Ali Akbar Arefijamaal [37] were study some properties of K-frames and introduced the K-frame multipliers.

In [3] A. Rahimi, Sh. Najafzadeh and M. Nouri were developed the concept of controlled K-frames. Some properties of K Dual frames are investigated by Vahid Reza, Mohammad Janfada and Rajabali KamyabiGol [66].

A.Najati and A. Rahimi [2] have developed the generalized frame theory and introduced methods for generating g-frames of a Hilbert space.

K-frames are more general than ordinary frames in the sense that the lower frame bound only holds for the elements in the range of K. Dingli Hua and Yongdong Huang [12] are proposed for construction methods for K-g-frames.

In [26] J.I.Giribet, A.Maestripieri, F.MartnezPeria, P.G. Massey are proposed definition of frames for Krein spaces and motivated the idea of J-frame for Krein space which extends the notion of J-orthonormal bases of Krein spaces. The basic concepts of Krein Space and hyper normal operators on Krein spaces are studied by Kevin Esmeral, Osmin Ferrer, Jorge Jalk and Boris Lora Castro [32]. Kevin Esmeral, OsminFerrer [32] have stated a definition of frames in Krein spaces and complete characterization is given by comparing them to frames in the associated Hilbert space.

ORGANIZATION OF THESIS

Chapter-1 (Introduction) is an introduction chapter and gives brief introduction about my research work. Also, we presented some basic concepts and related results from linear spaces, Hilbert spaces. The concepts and results given here can be found in [14] and gives basic definitions and concepts in Operator theory.

Chapter 2 (Literature Survey) Presents survey of the different approaches of Frame Theory available in the Literature and **Frame theory in Hilbert spaces** gives basic definitions in frames and frame operators with examples.

In this chapter we also presented some background information on Bessel sequences, Riesz basis, frames and types of frames for Hilbert spaces, also we provide definitions of frame operator, Analysis operator, Synthesis operators, Dual frame and Alternative dual frame. And this section contains some basic theorems and examples of frames and frame operators. The concepts and examples of frames on finite dimensional Hilbert spaces H_N are discussed.

Finally, we have given some basic definitions and concepts of g- frames, types of g-frames and we obtain some characterizations of g-frame operator.

In Chapter 3 (K- Frames) the definitions of different types of K- frames and examples of K- frames are given. Also, we study the frame operators of K-frames, and some results on K-frame operators are established.

In this chapter we presented some results on K-frames by using an injective closed range operator $K \in B(H)$.

We have given the definitions of 2K- frames, with the help of this we have proved, if $\{f_j\}$ is K-frame and $\{g_j\}$ is a 2K-frame then $\{f_j + g_j\}$ is a K-frame for H.

Also results on finite K-frames in finite dimensional Hilbert spaces are studied. The properties of eigen values of K-frame operator were discussed.

A K-g-frame is a generalization of a g-frame. It can be used to reconstruct elements from the range of a bounded linear operator K in Hilbert spaces. K-g-frames

have a certain advantage compared with g-frames in practical applications. **In Chapter 4 (K-g-frames**) we presented the definition and examples of K-g frames. Finally, by means of the methods and techniques in frame theory, several results on K-g-frames are obtained.

Results on K-frames have been proved through operator-theoretic results on quotient of bounded operators by G. Ramu and P.Johnson [17]. In this Chapter some of the theorems and propositions discussed in [17] are extended to K-g- frames.

Chapter 5 (Frames in Krein spaces) gives basic concepts of Krein spaces with examples. Definitions of frames and operator frames in Krein spaces with examples are studied.

Bibliography – gives the list of journals and conference papers and text books referred.

Contribution to work – Finally we included the list of contributions to the work by listing the papers published in national and international journals.

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CONTRIBUTION TO WORK:

Towards the contribution to work the following papers were published in journals.

1. “Operators Preserving K-g- frames”, Turkish Journal of Computer and Mathematics Education. Vol. 11 No. 03 (2020).1462-468, e-ISSN: 1309- 4653.
2. “Finite K-frames in Hilbert Spaces”, Journal of Mathematical and Computational Sciences, 11(2021), No.5, 6468-6480. ISSN No: 1927-5307.
3. “A Brief Study of Frames in Krein Space”, Journal of Xi’an University of Architecture & Technology, Volume XII, Issue V,2020, ISSN No. 1006-79.
4. “Some Results on 2K-frames in Hilbert spaces”, International Journal of Advanced Science and Technology. Communicated